Bright solitons from defocusing nonlinearities

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Part 1: Introduction

A commonly known principle of the soliton theory is that **self-focusing nonlinearity** is necessary to create **bright solitons** in uniform media.

Alternatively, stable **gap solitons** of the **bright** type can be supported by **self-defocusing nonlinearity** in the combination with a **periodic potential** (it may represent an **optical lattice** in BEC, or a **grating** or **photonic crystal** in optical media):

Nevertheless, it was considered obvious that the *self-defocusing nonlinearity* alone, unless combined with a *linear potential*, could not support *bright solitons*.

The main point of this talk is to demonstrate that this seemingly obvious “prohibition” can be *circumvented*. It is possible to support *stable bright solitons* in the $D$-dimensional space by a *purely self-defocusing* spatially modulated nonlinearity if its *local strength* grows fast enough at $|r| \to \infty$, namely, at any rate faster than $|r|^D$. 
Publications on the topic of this talk:


Related publications:


The structure of the talk:

Part 2: The model and analytical results for the steeply modulated local nonlinearity
Part 3: Numerical results for the same model with the steeply modulated local nonlinearity
Part 4: Analytical and numerical results for the model with mild modulation of the local nonlinearity
Part 5: A two-component model
Part 6: A generalization for a model with nonlocal repulsive interactions
Part 7: Conclusions
Part 2:
The model and analytical results for the model with steeply modulated local nonlinearity
To introduce the concept, we start with the following 3D NLS equation:

\[ i \frac{\partial q}{\partial \xi} = -\frac{1}{2} \nabla^2 q + \sigma(r) |q|^2 q. \]

Here \( \xi \) is the propagation distance (in optics) or time (in BEC), \( r = (\eta, \zeta, \tau) \) is the set of transverse coordinates, \( \nabla^2 = \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \tau^2} \) is the transverse diffraction/dispersion operator, \( r^2 = \eta^2 + \zeta^2 + \tau^2 \), and the local strength of the defocusing nonlinearity, growing at \( r \to \infty \), is taken, for the time being, in the form of the anti-Gaussian, with the pre-exponential factor:

\[
\sigma(r) = \left( \sigma_0 + \frac{1}{2} \sigma_2 r^2 \right) \exp \left( \frac{r^2}{2} \right), \text{ with } \sigma_0, \sigma_2 \geq 0.
\]
Such a steep (anti-Gaussian) growth at $r \to \infty$ is not a necessary condition for the existence of stable bright solitons with a finite (convergent) norm in the model. It is sufficient to adopt the following asymptotic form of the growth of the local strength of the nonlinearity at $r \to \infty$:

$$
\sigma(r) = \text{const} \cdot r^{D+\varepsilon},
$$

with any positive $\varepsilon$, where $D$ is the spatial dimension.

In BEC, spatially modulated profiles of the nonlinearity can be induced, via the Feshbach resonance, by a nonuniform external magnetic or laser field:
In the case when the resonant dopant is used (in optics), the trend to the divergence of the local nonlinearity coefficient at $r \to \infty$ may be achieved not necessarily through the increase of the density of the dopant, but rather by gradually tuning the dopant to the exact resonance at $r \to \infty$. In a similar way, the divergence may be achieved by means of tuning the Feshbach resonance in BEC.
Stationary solutions with a real propagation constant, $b$, are looked for in the usual form:

$$q(r, \xi) = \exp(\, ib\xi\,) w(r),$$

with real propagation constant $b$, and function $w$ satisfying the stationary equation:

$$-bw = -\frac{1}{2} \nabla^2 w + \sigma(r) |w|^2 w.$$
The model with the \textit{anti-Gaussian} (steep) spatial modulation is chosen as the first example because it admits a particular \textit{exact solution} for the \textit{fundamental solitons} in the form of a \textit{Gaussian}, at a \textit{single value} of propagation constant \( b \), for \textit{any dimension} \( D \):

\[
q(r, \xi) = \left( \frac{1}{2\sqrt{\sigma_2}} \right) \exp \left( ib\xi - \frac{r^2}{4} \right),
\]

\[
b = -(1/4) \left( D + \frac{\sigma_0}{\sigma_2} \right).
\]
This solution does not exist at $\sigma_2 = 0$, but in that case it is possible to find exact solutions for a 1D dipole (twisted) soliton, and a 2D vortex soliton with topological charge $m = 1$:

$$q(r, \xi) = \frac{1}{2\sqrt{2}\sigma_0} r \exp \left( ib\xi + i\phi - \frac{r^2}{4} \right)$$

(the azimuthal coordinate, $\phi$, appears only at $D = 2$), with propagation constant $b = -(1/2)(1 + D/2)$.

Recall that the profile of the modulation of the self-defocusing nonlinearity in this case ($\sigma_2 = 0$) is $\sigma(r) = \sigma_0 \exp\left(\frac{r^2}{2}\right)$. 
In the general case (for arbitrary values of the propagation constant, \( b < 0 \)), a family of *approximate analytical solutions* can be constructed by means of the *variational approximation*.

The corresponding *ansatz* is taken in the simplest *Gaussian* form, with amplitude \( A \),

\[
q(\xi, r) = A \exp \left( ib\xi - \frac{r^2}{4} \right).
\]

The norm of the ansatz is \( U \equiv \int |q(r)|^2 \, dr = (2\pi)^{D/2} A^2 \).

Here we again take the simplest version of the model, i.e.,

\[
i \frac{\partial q}{\partial \xi} = -\frac{1}{2} \nabla^2 q + \sigma_0 \exp \left( \frac{r^2}{2} \right) |q|^2 q.
\]
The variational approximation yields the following relation between the **norm** and **propagation constant** of the soliton:

\[ U = - (2\pi)^{D/2} (b + D/4) \]. Comparison of this prediction with numerical findings will be presented below.

In the **general case**, it is possible find an **analytical asymptotic expression** for the soliton's **tail** at \( r \to \infty \), which is **universal**, as it **does not depend** on the dimension \( (D) \) and propagation constant \( b \), nor on the type of the soliton (**fundamental, vortical**, etc.):

\[ q \approx \exp(ib\xi)(r/2^{3/2})\exp(-r^2/4) \].

Note that this asymptotic expression is obtained **keeping the nonlinear term** in the equation, i.e., the equation of this type is **nonlinearizable** at \( r \to \infty \), for the **decaying soliton tails**. In fact, it is the **nonlinearizability** of the tail which makes the existence of the solitons possible in this type of the nonlinear media.
Particular exact solutions in 1D can be also found for other forms of the nonlinearity modulation, with the local strength growing \textit{exponentially} (rather than as the \textit{anti-Gaussian}) at $r \to \infty$.

The \textbf{fundamental soliton} can be found for the following 1D equation:

\begin{align*}
    i \frac{\partial q}{\partial \xi} &= -\frac{1}{2} \frac{\partial^2 q}{\partial \eta^2} + (a + \sinh^2(\eta)) |q|^2 q, \\
    \text{for } a &< 1: \\
    q &= \frac{e^{ib\xi}}{\sqrt{1-a}} \text{sech}(\eta), \text{ with } b = -\frac{1+a}{2(1-a)}. 
\end{align*}
At $a=1$, when the equation takes the form of

$$i \frac{\partial q}{\partial \xi} = -\frac{1}{2} \frac{\partial^2 q}{\partial \eta^2} + \cosh^2(\eta) |q|^2 q,$$

the above exact solution for the fundamental soliton does not exist. However, in this case an exact solution can be found for a 1D dipole (twisted) soliton:

$$q = e^{ib\xi} \frac{\sqrt{3} \sinh(\eta)}{\cosh^2(\eta)}, \text{ with } b = -\frac{5}{2}.$$
Part 3:
Numerical results for the model with the anti-Gaussian (steep) modulation of the local nonlinearity

Recall the form of the model:

\[ i \frac{\partial q}{\partial \xi} = -\frac{1}{2} \nabla^2 q + \sigma_0 \exp \left( \frac{r^2}{2} \right) |q|^2 q. \]

In the 1D case, all the fundamental solitons are stable. The higher-order 1D solitons with \( k \) nodes are also completely stable for \( k = 1, 2 \). Instability regions appear for \( k \geq 3 \). Stable higher-order 1D solitons were found even for \( k = 10 \).
Stationary profiles \( w = |q| \) and stability of the fundamental and higher-order 1D solitons with \( k \) nodes. Red curves: the nonlinearity-modulation Gaussian profile, \( \sigma(\eta) \). In (c), variational and numerical \( U(b) \) curves completely overlap for fundamental solitons, \( k = 0 \) (the green segments for \( k = 4 \) are unstable). In (d), gray regions are unstable for \( k = 5 \).
Unstable solitons spontaneously transform into breathers, which remain tightly localized. Typical examples of the evolution of stable and unstable 1D solitons are displayed here (top row), along with examples of oscillations of kicked 1D stable solitons (bottom row; the kick is $\theta = 1.5$).
Profiles and **stability** of 2D solitons with vorticity $m$. All vortices with $m = 1$ are **stable**. Unstable vortices appear starting from $m = 2$. In (b), $m = 2$. In (c), variational and numerical $U(b)$ curves **completely overlap** for fundamental solitons, $k = 0$. In (d), gray regions are **unstable** for $m = 2$. 
Examples of the **stable** evolution of a 2D vortex with $m = 2$ (a), and **spitting** of **unstable** vortices with $m = 2$ (b) and $m = 3$ (c) into *rotating sets* of unitary vortices.
In the 3D model (which cannot be applied to optical media, but may be relevant to BEC), the family of fundamental solitons is completely stable. An example of the relaxation of a perturbed stable fundamental 3D soliton with $b = -10$ is presented here. Snapshots are displayed at $\xi = 0, 300, 600$. 
Part 4: Analytical and numerical results for 1D and 2D solitons in the model with the mild modulation of the local nonlinearity
The same model, but with a *slow growth* of the local nonlinearity at $r \to \infty$:

$$i \frac{\partial q}{\partial \xi} = -\frac{1}{2} \nabla^2 q + (1 + r^\alpha)|q|^2 q,$$

with $\alpha > 0$. In the 1D case, $r$ is replaced by $|\eta|$.

In this model, *fundamental solitons* can be found, for any $D$, by means of the Thomas-Fermi (TF) approximation, dropping the Laplacian:

$$|q|^2 \approx -b/(1 + r^\alpha).$$

The corresponding norm of the soliton is

$$U_{\text{TF}} = \frac{2\pi^D |b|}{\alpha \sin(\pi D/\alpha)}.$$

This formula shows that the fundamental solitons exist for $\alpha > D$, i.e. (as mentioned above), if the local strength of the self-defocusing grows at $r \to \infty$ *at any rate faster than* $r^D$. 
Examples of 1D and 2D fundamental and vortical solitons. (a) 2D, $\alpha = 5$, $b = -10$, for vorticities $m = 0, 1, 2$; (b) 2D, $m = 1$, $b = -5, -10, -20$ (curves 1, 2, 3); (c) 1D, $b = 10$, $\alpha = 5$; $k = 0$ – fundamental, $k = 2$ – second-order solitons. The TF profiles for the 2D and 1D fundamental solitons are indistinguishable from their numerical counterparts.
Numerical and analytical results for families of solitons. (a): 2D, $\alpha = 5$; (b): 2D, $b = -10$; (c): 1D, $b = -10$ [red curves in (b) and (c) display the TF predictions]. In the exact agreement with TF, $\alpha_{cr} = D$. 
All the fundamental 1D and 2D solitons are stable. Examples of the evolution: unstable vortices with $m = 1, \alpha = 3.5, b = -3$ (a) and $m = 2, \alpha = 4, b = -20$ (b) – transformation into a fundamental soliton in (a), and into a stable vortex with $m = 1$ in (b); (c) **self-healing** of two stable vortices with $m = 1, \alpha = 5$, or $m = 2, \alpha = 10$. 
Part 5:  
A two-component 1D model

The system of two 1D equations with the steep anti-Gaussian modulation, coupled by XPM terms with coefficient $C$:

$$i \frac{\partial q_{1,2}}{\partial \xi} = - \frac{1}{2} \frac{\partial^2 q_{1,2}}{\partial \eta^2} + \exp(\eta^2)(|q_{1,2}|^2 + C |q_{2,1}|^2)q_{1,2}. $$
Examples of solitons with **overlapping** (a,b) and **separated** (c,d) centers ($w_{1,2} = |q_{1,2}|$) of the components. $C = 2$, $b_1 = -5$ in all panels, and $b_2 = -8$ in (a), -3.1 in (b), -5 in (c), -7.2 in (d). The soliton is **unstable** in (a), and **stable** in the other cases.
The most essential property of the two-component solitons is the transition from the *overlapping* to *separated* components in *symmetric* solitons, with $b_1 = b_2 \equiv b$. The transition leads to *destabilization* of the overlapping solitons.

The transition can be predicted analytically by means of a *variational approximation*, based on *ansatz*

$$q_{1,2} = A \left(1 \pm \zeta \eta - \frac{1}{2} \eta^2 \right) \exp \left( i b \xi - \frac{1}{2} \eta^2 \right),$$

where $2 \zeta$ is the separation between centers of the two components.

The *result* is that solitons with the separated components exist at

$$C > C_{\text{thr}} = \frac{b-1/4}{b+3/4}, A^2 = -\frac{b+1/4}{C+1}.$$ 

For instance, at $b=-5$ the formula yields $C_{\text{thr}} = 1.23$, while the numerical result is $C_{\text{thr}} = 1.19$. A "naive" result for the uniform space is $C_{\text{thr}} \equiv 1$. 
Part 6:
1D solitons in the model with the spatially modulated *nonlocal* self-defocusing nonlinearity

The model:

\[ iu_z + \frac{1}{2} u_{xx} - mu = 0, \]
\[ m - dm_{xx} = \sigma(x) |u|^2. \]

Here \( d \) is the squared correlation length of the nonlocal response, \(-m\) is a local perturbation of the refractive index, due to heating by the light wave, assuming that the **local heating decreases** the local value of refractive index. \( \sigma(x) \) is the local density of the absorptive dopant.
Stationary solutions with propagation constant $b$ and integral power (norm) $P$ are looked for as

$$U(x, z) = e^{ibx}U(x), \ m = M(x),$$

with $U(x)$ and $M(x)$ obeying equations

$$-bU + \frac{1}{2}U'' - MU = 0,$$

$$M - dM'' = \sigma(x)U^2,$$

and $P \equiv \int_{-\infty}^{+\infty} U^2(x)dx$. 
An example of an **exact analytical solution** can be obtained for \( \sigma(x) = \sigma_{-2} \cosh^2(ax) + \sigma_0 - \sigma_2 \text{sech}^2(ax) \), with \( \sigma_0 = \frac{2}{3} \left( 1 - \frac{1}{4da^2} \right) \sigma_2 \), namely

\[
U(x) = A \text{sech}(ax), \quad M(x) = M_0 - M_2 \text{sech}^2(ax).
\]

The coefficients of the exact solution are

\[
A^2 = \frac{6da^4}{\sigma_2}, \quad M_0 = \frac{6\sigma_{-2}}{\sigma_2} da^4, \quad M_2 = a^2,
\]

\[
b = \frac{a^2}{2} \left( 1 - \frac{12\sigma_{-2}}{\sigma_2} da^2 \right).
\]
Another example, of an **exact twisted (dipole-shaped)** solution, is available for \( \sigma(x) = \sigma_2 \cosh^2(ax) + \sigma_0 \):

\[
U(x) = B \sinh(ax) \operatorname{sech}^2(ax), \quad M(x) = M_0 - M_2 \operatorname{sech}^2(ax),
\]

with \( \sigma_2 = \frac{1}{3} \left( 1 + \frac{1}{4da^2} \right) \sigma_0 \), and coefficients

\[
B^2 = \frac{18da^4}{\sigma_0}, \quad M_0 = \frac{18\sigma_2}{\sigma_0} da^4, \quad M_2 = 3a^2,
\]

\[
b = -\frac{a^2}{2} \left( \frac{1}{2} + 3da^2 \right).
\]
The *power-vs.-propagation-constant* curve for the *family* of fundamental solitons at $d = a = \sigma_2 = 1$. The **red dot** indicates the particular exact solution:
The shape of the exact soliton solution and its stability in direct simulations:
The *power-vs.-propagation-constant* curve for the *family* of twisted solitons at $d = a = \sigma_2 = 1$. The *red dot* indicates the particular exact solution:
The shape of the exactly found soliton and its \textit{stability} in direct simulations:
Part 6: Conclusion

The local modulation of the purely self-defocusing nonlinearity, with the local strength diverging at $r \to \infty$, can support stable fundamental solitons in the space of any dimension, as well as multipoles and vortices in 1D and 2D. The necessary growth rate of the local nonlinearity strength is anything faster than $r^D$. 
The setting can be implemented in **optics**, using *resonant dopants* with a nonuniform density, and in **BEC** by means of the *Feshbach resonance* controlled by nonuniform external fields.

Many results have been obtained in the analytical form – **exact**, in some particular cases, and, in the general case, by means of the *variational* and *Thomas-Fermi approximations*. 
Stable 1D solitons, of the fundamental and twisted types, were also found in the model with the nonlocal self-defocusing nonlinearity, whose strength grows fast enough towards $|x| \to \infty$. 