Exact Solution of Ising Model on a Small-World Network

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Outline

- The model
- The solution
- Thermodynamics
- Finite size scaling behavior
- Discussion and conclusions
Shortest distance, averaged over pairs:

- $O(N)$ for ordered networks
- $O(\ln N)$ for networks with $pN$ shortcuts ($p > 0$)
Building the network

Start with ring of $N$ connected points (periodic bc). For each site, with a probability $p$, we make a connection with a randomly chosen site.
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The model

\[ H = -J \sum_{i=0}^{N-1} \sigma_i \sigma_{i+1} - I \sum_{(ij) \in S} \sigma_i \sigma_j = - \sum_{(i,j)} J_{ij} \sigma_i \sigma_j \]

- Short range links: interaction strength \( J \).
- Long range shortcuts: interaction strength \( I \).
- \( S \) is the set of pairs with long shortcuts (\( \#S = pN \)).
Previous results

Simulations (Hong et. al 2002, Herrero 2002)

- $T_c$ finite for any $p > 0$;
- Mean-field critical behavior;

Analytical results (Barratt and Weigt 2000, Gitterman 2000)

- Agreement over mean-field behavior;
- Gitterman predicts no order for $p < 1/2$.

Bethe lattice representation (Dorogvtsev, Goltsev and Mendes, 2002)
If it is mean-field, does it have an analytical solution?
Spin → Bond transformation

\[ Z = \text{Tr}_{\{\sigma\}} \exp(\beta \sum_{(i,j)} J_{ij} \sigma_i \sigma_j) = \text{Tr}_{\{\sigma\}} \prod_{(i,j)} \exp(\beta J_{ij} \sigma_i \sigma_j) \]

Identity \((\sigma_i = \pm 1)\)

\[
\frac{\exp(\beta J_{ij} \sigma_i \sigma_j)}{\cosh(\beta J_{ij})} = 1 + \sigma_i \sigma_j \tanh(\beta J_{ij}) = \sum_{b_{ij}=0,1} (\tanh(\beta J_{ij}) \sigma_i \sigma_j)^{b_{ij}}
\]
\[ Z = \left( \prod_{(i,j)} \cosh \beta J_{ij} \right) \sum_{b} \sum_{\sigma} \prod_{(i,j)} (\sigma_i \sigma_j \tanh \beta J_{i,j})^{b_{ij}} \]

Sum over \( \sigma_k \), for a given bond configuration:

- factor 2, if \( \sum_j b_{kj} \) is even;
- 0, if \( \sum_j b_{kj} \) is odd;
\[ Z = Z_0 \text{Tr}'_{\{b\}} \prod_{(i,j)} (\tanh(\beta J_{i,j}))^{b_{ij}} \]

- \[ Z_0 = 2^N \prod_{(i,j)} \cosh \beta J_{i,j} \]
- \textbf{Tr'}: Trace restricted to the configurations of } b' \text{'s with } \sum_j b_{k,j} \text{ even, for all } k.
Example: Ising 1D

Each site \( i \) has 2 links: surviving bond configurations must have \( b_{i-1,i} = b_{i,i+1} \). Only two possible bond configurations contribute to \( Z \):

\[
\begin{align*}
\text{(a)} & : & b_{ij} = 0 & \forall i, j \\
\text{(b)} & : & b_{ij} = 1 & \forall i, j
\end{align*}
\]

\[
Z = 2^N \cosh^N(\beta J) (1 + \tanh^N(\beta J))
\]

\[\uparrow \quad \uparrow\]

(a) \quad (b)
**This model**

**Restriction:** at most 1 shortcut per site.

Fix the $b$ values of shortcuts and the $b$ value of one short range link and all the $b'$s are determined.
Partition function

\[ Z = Z_0 c_I^{N_b} \sum_{b_0} \sum_{\{b_1, \ldots, b_{pN}\}} t_J^L t_I^M \]

With \( c_I = \cosh(\beta I) \), \( t_J = \tanh(\beta J) \), \( t_I = \tanh(\beta I) \) and

- \( L[b] \) Number of short range links with \( b = 1 \);
- \( M[b] \) Number of shortcuts with \( b = 1 \).

\[ Z = Z_0 c_I^{N_b} \sum_{b_0} \sum_{M,L} e^{\ln \Omega(M,L) + L \ln t_J + M \ln t_I} \]
Number of states

Two models solved

A

B
Number of states

Number of shortcuts, $N_b = pN$

Choose $0 \leq 2M \leq 2N_b$ points (filled).

$L/d = l_2 + l_4 + \cdots + l_{2M}$ \quad ($l_i$, integer, $d = 1/2p$)

$(N - L)/d = l_1 + l_3 + \cdots + l_{2M+1}$

$$C_{M-1}^{L/d-1} \times C_M^{(N-L)/d}$$
Number of states

Pick bonds, not sites! Normalizing factor:

- $C_{2M}^{2pN}$ → number of ways to pick $2M$ sites;
- $C_M^{pN}$ → number of ways to pick $M$ bonds;

$$
\Omega(M, L) = \frac{C_M^{pN}}{C_{2M}^{2pN}} \times C_M^{2p(N-L)} C_{M-1}^{2pL-1}
$$
Thermodynamical behaviour

Intensive variables

\[ n = \frac{M}{pN} \]
\[ l = \frac{L}{N} \]

\[ \ln \Omega(M, L) + L \ln t_J + M \ln t_I = N \left( s(l, n) + l \ln t_J + pn \ln t_I \right) \]
\[ + O(\ln N) \]

\[ \lim_{N \to +\infty} \implies \text{steepest descent!} \]
\[ \ln Z = N \ln z_0 + pN \ln c_I + pN f_1 \]
Critical temperature ($T_c$)

Maximum exits at a temperature given by

$$(2t_I + 1) t_f^d = 1$$
Asymptotic behaviour

\( I/T_c \gg 1 \)

\[
T_c = \frac{2J}{\ln \left( \frac{1}{p \ln 3} \right)}
\]

\( I/T_c \ll 1 \)

\[
T_c = \frac{2J}{\ln \left( \frac{J}{pI \ln (J/(pI))} \right)}
\]
Partition Function

\[ \ln Z = N \ln z_0 + pN \ln c_I + pN f_1 \]

\[ f_1 = \begin{cases} 
\ln \left( \frac{4(1-l)^2(1-n)}{(2-2l-n)^2} \right) & T < T_c \\
0 & T > T_c 
\end{cases} \]

\[ n \propto T_C - T \]
\[ l \propto T_C - T \]
Specific heat
Finite size scaling (FSS)

Does this solution give the FSS behaviour?

\[ \frac{C_N(T)}{C_\infty(T)} \quad \text{vs} \quad \zeta \equiv tN^{1/2} \]
Analytical Result

\[ Z_N = \exp[-\beta N(f_0 + f_a)]Z_{\text{FSS}} \]

\[ Z_{\text{FSS}} = \frac{N}{2\pi p} \int_0^{\infty} dl \int_0^2 du g(l,u) \exp(pNh(l,u)) \]

\[ h(l,u) \approx -c_1(l-l^*)^2 - c_2(u-u^*)^2 \]

\[ Z_{\text{FSS}}(\zeta) \approx \frac{\sqrt{\pi N^{1/4}}}{k_1} \int_0^{\infty} \frac{e^{-(x+k_2\zeta)^2}}{\sqrt{x}} \, dx. \]
Bethe Lattice Approach

Network is locally a Bethe lattice (loops $\sim \mathcal{O}(\log N)$)

\[
\langle \sigma_0 \sigma_{(L', M)} \rangle N_s(L', M) = e^{-\kappa(x, T)(L' + M)} , \quad x \equiv \frac{L'}{L' + M}
\]

\[\kappa_{\text{min}}(x) \to 0 \text{ at } T_c!\]
Bethe Lattice Approach

\[ \langle \sigma_0 \sigma(L, M) \rangle \sim \exp\left( -\frac{L + M}{\xi_{1D}} \right) \]

\[ L + M \sim \ln N \]

\[ \langle \sigma_0 \sigma(L, M) \rangle \sim N^{-1/\xi_{1D}} \]
Conclusions

- Mean-field ferromagnetic transition;
- Finite size scaling not governed by $\xi$ (violation of hyperscaling);

Lacking

- order parameter, $\chi$, finite field;
- any way of having $lcd < d_{eff} < ucd$. 